Introduction to Algorithm Design and Analysis

Chapter 1

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What is an algorithm ?

- Informally an *algorithm* is a well-defined computational procedure comprising a sequence of steps for solving a particular problem.

- Donald Knuth identifies the following five characteristics of an algorithm:
  - *Input:* there are zero or more quantities which are externally supplied.
  - *Output:* at least one quantity is produced.
  - *Finiteness:* terminates after a finite number of steps for all possible inputs.
  - *Definiteness:* each step must be clear and unambiguous.
  - *Effectiveness:* every step must be sufficiently basic that it can in principle be carried out by a person using only pencil and paper.

Computational problems

- A *well-defined computational problem* is a pair $P = (I,O,R)$ such that:
  - I is a specification of the set of allowed inputs
  - O is a specification of the set of outputs
  - R is a specification of the desired relation between an input and an output.
- A *problem instance* is given by a specific input $i \in I$.
- Example – the *sorting problem*:
  - *Input:* a sequence $(a_1, a_2, \ldots, a_n)$, a_i, $1 \leq i \leq n$, elements of an ordered set
  - *Output:* a permutation (reordering) $(b_1, b_2, \ldots, b_n)$ of the input sequence such that $(b_1 \leq b_2 \leq \ldots \leq b_n)$.

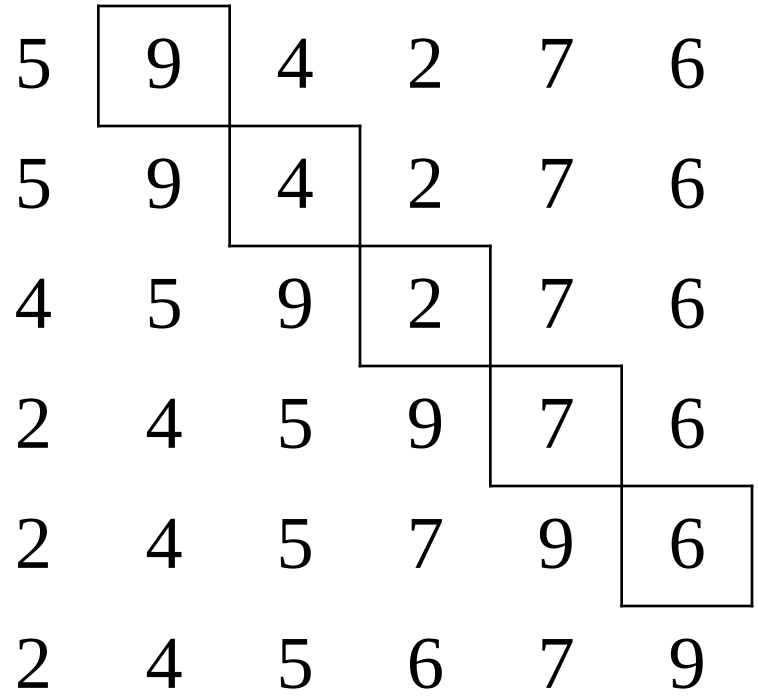
Insertion sort – explanation

- We consider our array *A* with indexes from 1 to *n* = *length*[*A*]
- The array is split in two parts:
  - 1st part with lower indexes from 1 to *j*-1 (*j* ≥ 2) is *already sorted*. Initially it contains only the first element *A*[1].
  - 2nd part with upper indexes from *j* to *n*. This is *not touched*.
- In each step move the first (smallest index) element from the 2nd part to the 1st part such that 1st part is kept sorted.
- Each step assumes:
  - Create an "empty place" in position *j*. *A*[*j*] is "saved" into variable *key*.
  - Search the position where to insert *key* in 1st part. This is done exactly like searching the place of a new book on a book shelf where the books are alphabetically sorted according to the author name.
    - Start comparing *key* with the "highest" *A*[*i*] for *i* = *j*-1. If *key* < *A*[*i*] then shift right *A*[*i*] with one position. Shifting means moving the "empty place" to the left.
    - Stop shifting when you find an *i* for which *key* ≤ *A*[*i*] or you reached the end of the 1st part (*i* = 0).
  - Copy the key into the found position represented by *i*+1.

Insertion sort – algorithm

INSERTION-SORT(A)

1. **for** $j \leftarrow 2, length[A]$ **do**
2. $\quad key \leftarrow A[j]$
3. $\quad \triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \ldots j-1]$
4. $\quad i \leftarrow j-1$
5. $\quad$ **while** $i > 0$ and $A[i] > key$ **do**
6. $\quad\quad A[i+1] \leftarrow A[i]$
7. $\quad\quad i \leftarrow i-1$
8. $\quad A[i+1] \leftarrow key$



Issues in studying algorithms I

- *Correctness*
  - An algorithm is *correct* iff for all problem instances $i \in I$ it terminates and produces the correct output $o \in O$ (i.e. the pair (i,o) satisfies R).
  - An *incorrect* algorithm either does not terminate or terminates and produces a wrong output for at least one input.

- *Design*
  - Various design techniques that often yield good algorithms have been established: recursion, divide and conquer, dynamic programming, a.o.

Issues in studying algorithms II

- *Specification*
  - An algorithm can be specified in various ways using: natural languages, informal pseudo-codes or formal specifications, programming languages.

- *Analysis*
  - Evaluates the amount of resources (processor time, memory) required by an algorithm.

- *Testing*
  - Debugging (error correction) and profiling (deriving performance profiles of algorithms).

Pseudo-code conventions

- Indentation indicates block structure.
- Conditional and looping statements are similar to Pascal.
- The symbol $\triangleright$ starts a comment.
- Multiple assignments $i \leftarrow j \leftarrow e$ have a semantics similar to C.
- Variables are local unless otherwise is explicitly specified.
- Indexing is specified as in C and Pascal: $A[i]$. A sub-array of A is specified as $A[i \ldots j]$.
- Compound data are organized as objects or structures. For an object x the field f is specified as $f[x]$.
- An object variable is a reference (or pointer) to the object (similar to Java).
- Parameters are passed by value.

Algorithm analysis

- Estimates the amount of resources required by an algorithm:
  - Execution time
  - Memory
- We assume the execution model of the Random-Access Machine (RAM hereafter): one processor + memory + sequential execution.
- Execution time depends on:
  - *Input size* (i.e. 4 elements versus 10000 elements)
  - *Input itself* (the degree of sort-ness: input partially sorted). The input size is problem dependent: no.of elements, no.of bits, a.o.
- *Execution time* = no. of primitive operations or executed steps.
- A *step* should be independent of a specific computer. We assume that the time for executing a pseudo-code line on the RAM is constant.

Analysis of insertion sort

INSERTION-SORT(A)	Cost	Time
1. **for** $j \leftarrow 2, length[A]$ **do**	c_1	n
2. $key \leftarrow A[j]$	c_2	$n-1$
3. $\triangleright$ Insert $A[j]$ into the sorted sequence $A[1 \ldots j-1]$	0	$n-1$
4. $i \leftarrow j-1$	c_4	$n-1$
5. **while** $i > 0$ and $A[i] > key$ **do**	c_5	$\sum_{j=2}^{n} t_j$
6. $A[i+1] \leftarrow A[i]$	c_6	$\sum_{j=2}^{n}(t_j-1)$
7. $i \leftarrow i-1$	c_7	$\sum_{j=2}^{n}(t_j-1)$
8. $A[i+1] \leftarrow key$	c_8	$n-1$

Execution time: $T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n}(t_j-1) + c_7 \sum_{j=2}^{n}(t_j-1) + c_8(n-1)$

$n = length(A)$ and t_j is the number of executions of the test of the while loop in line 5.

Worst, average and best cases

Let $P = \langle I, O, R \rangle$ be a problem and A be an algorithm for solving P. Let $T(A(i))$ the execution time for executing A on instance $i \in I$. Taking into account the contents of the input data we can define three cases:

i) *Worst*. $T_w^A(n) = \sup\{T(A(i)) | i \in I, |i| = n\}$

ii) *Best*. $T_b^A(n) = \inf\{T(A(i)) | i \in I, |i| = n\}$

iii) *Average*. $T_a^A(n) = \frac{\sum_{i \in I, |i|=n} T(A(i))}{|\{i \in I, |i| = n\}|}$

Most often we prefer the worst case because:

- It is an upper bound for the execution time for all inputs
- It occurs frequently
- The average case is very often as bad as the worst case

Worst, average and best cases for insertion sort

Best case = the input array is already sorted, i.e. $t_j = 1$ for all $j = 2, 3, \ldots, n$. $T_b(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n-1) + c_8(n-1) = (c_1 + c_2 + c_4 + c_5 + c_8)n - (c_2 + c_4 + c_5 + c_8)$

Worst case = the input array is sorted the other way around, i.e. $t_j = j$ for all $j = 2, 3, \ldots, n$. $T_w(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5(n(n+1)/2 - 1) + c_6 n(n-1)/2 + c_7 n(n-1)/2 + c_8(n-1) = (c_5/2 + c_6/2 + c_7/2)n^2 + (c_1 + c_2 + c_4 + c_5/2 - c_6/2 - c_7/2 + c_8)n - (c_2 + c_4 + c_5 + c_8)$

Average case = we have to check half of the elements of $A[1 \ldots j-1]$, so $t_j = j/2$. $T_a(n)$ still is a quadratic function of n.

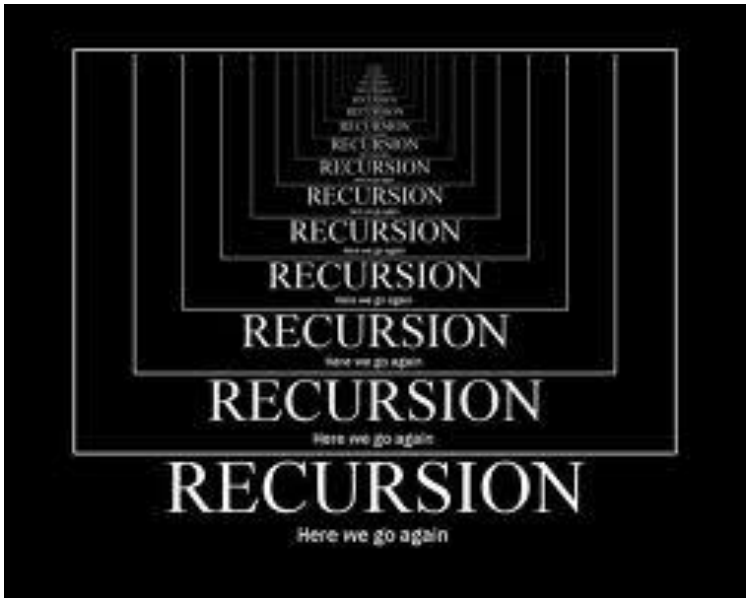
Asymptotic analysis

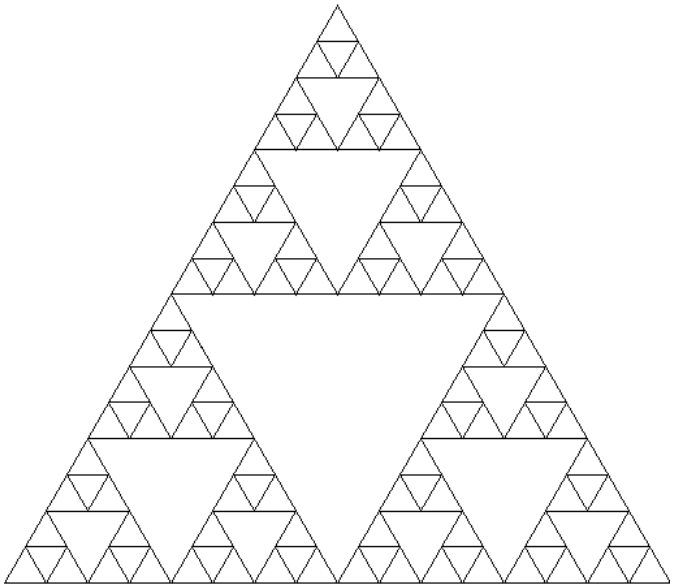
- In analyzing the execution time we have already made a *first assumption*:
  - the real cost of an instruction is independent of the underlying machine.
- Moreover, we have noticed that the execution time was a function $an^2 + bn + c$, so the values of the constants c_i can be abstracted away as well.
  - *Second assumption* = execution time is a polynomial of degree 2.
- In what follows we make a *third simplifying assumption*:
  - We shall only be interested in the growth of the execution time. This can be expressed with the Θ notation. Using this notation the worst case execution time of insertion sort is $\Theta(n^2)$.
  - The growth is determined by considering only the dominant term in the expression of $T(n)$ and then ignoring the a constant.
  - We can simply say that the execution time is quadratic.

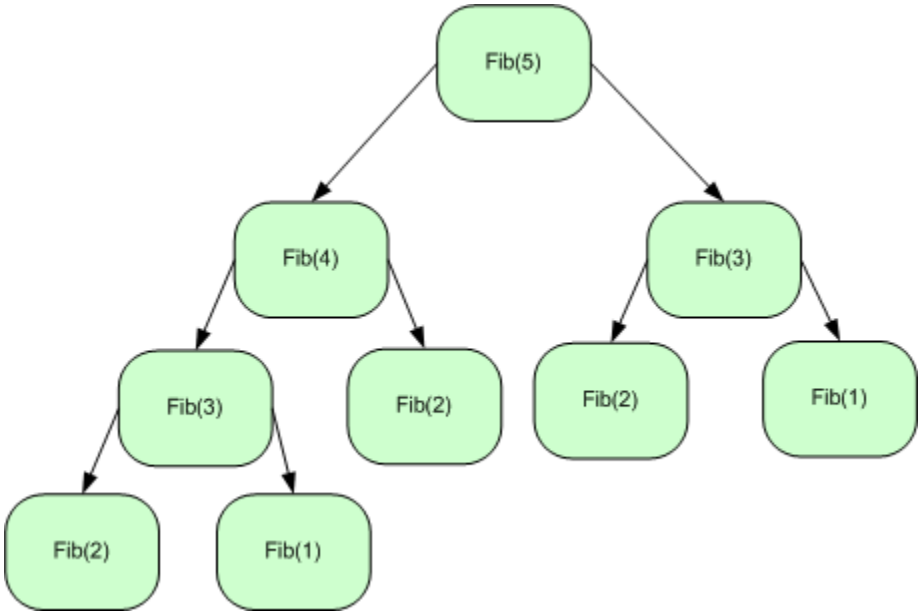
Recursion

- Recursion = defining a concept with a definition that refers directly or indirectly to the concept itself.
- Examples:
  - A *royal person* is either a monarch or a descendant of a royal person.
  - A *Canadian citizen* is either a person born in Canada, a person that obtained the citizenship after emigrating to Canada or a child of a Canadian citizen.
  - An *arithmetic expression* is either a constant, a variable, an arithmetic expression between parentheses or two arithmetic expressions separated by an operator.
- A *recursive definition* contains non-recursive definition rules and recursive definition rules. We have simple recursion, double recursion, depending on the number of references of the defined concept in a recursive definition rule.
- In programming we have *algorithm recursion* and *data recursion*.

Recursion illustrated







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Designing algorithms by divide and conquer

- An algorithm design technique that uses recursion is *divide and conquer*:
  - *Divide* the problem into smaller sub-problems.
  - *Conquer* the sub-problems by recursion if they are small.
  - *Combine* the solutions to the sub-problems into the solution of the original problem.
- We can apply divide and conquer to sorting:
  - *Divide* the input sequence into 2 sub-sequences.
  - *Conquer* the sub-sequences by sorting them recursively.
  - *Combine* the sorted sub-sequences into the final sorted sequence.
- Merging = combining 2 sorted sequences into a sorted sequence. If their total length is n then this takes $\Theta(n)$ time.
- In *merge sort* dividing is splitting and combining is merging.
- Note that other divide and conquer sorts are possible.

Merge sort

MERGE-SORT(A, p, r)

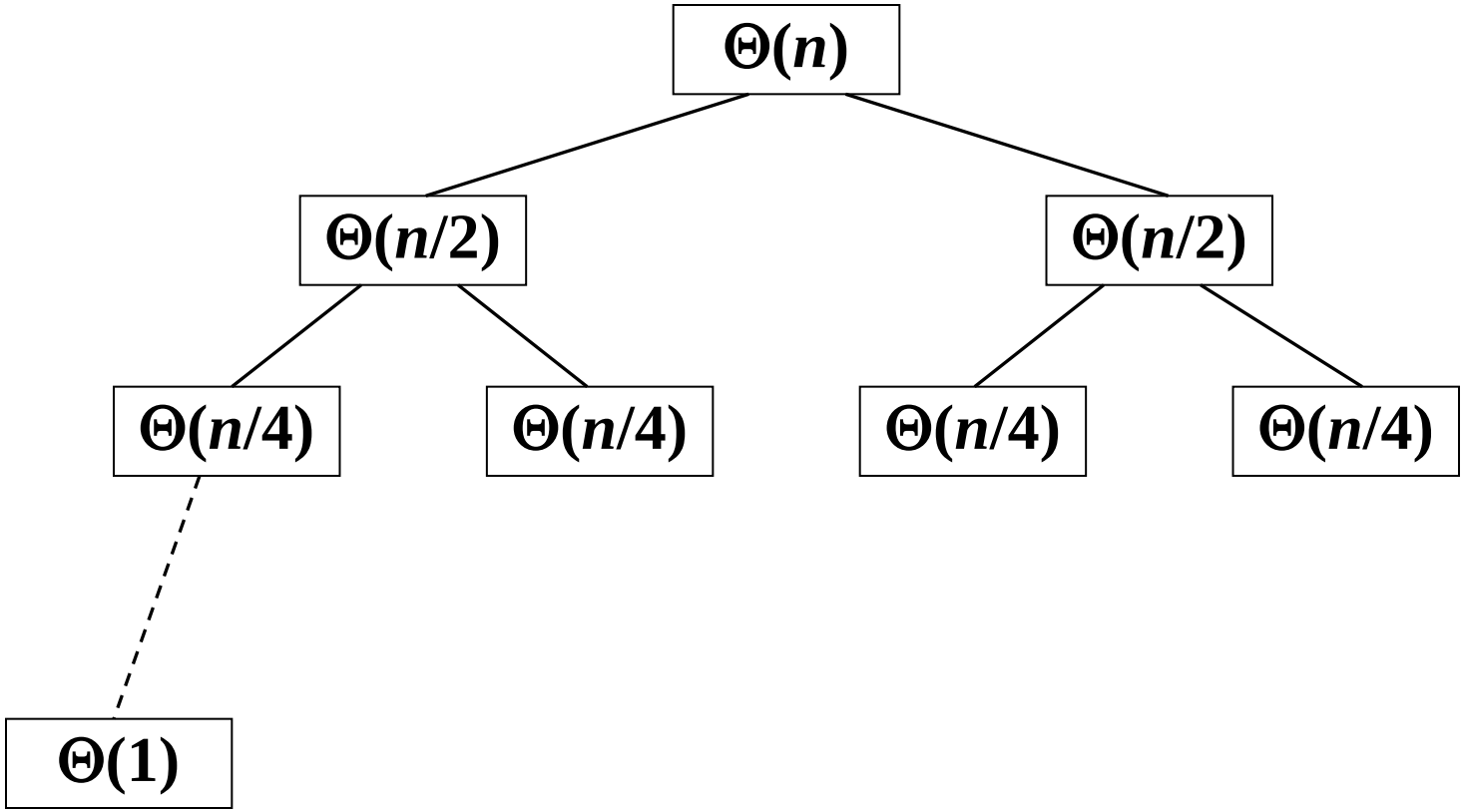
1. **if** $p < r$ **then**
2. $\quad q \leftarrow \lfloor (p + r)/2 \rfloor$
3. $\quad$ MERGE-SORT(A, p, q)
4. $\quad$ MERGE-SORT$(A, q + 1, r)$
5. $\quad$ MERGE(A, p, q, r)

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = 1 \\ 2T(n/2) + \Theta(n) & \text{if } n > 1 \end{cases}$$

Solving this recurrence we get $T(n) = \Theta(n \lg n)$.

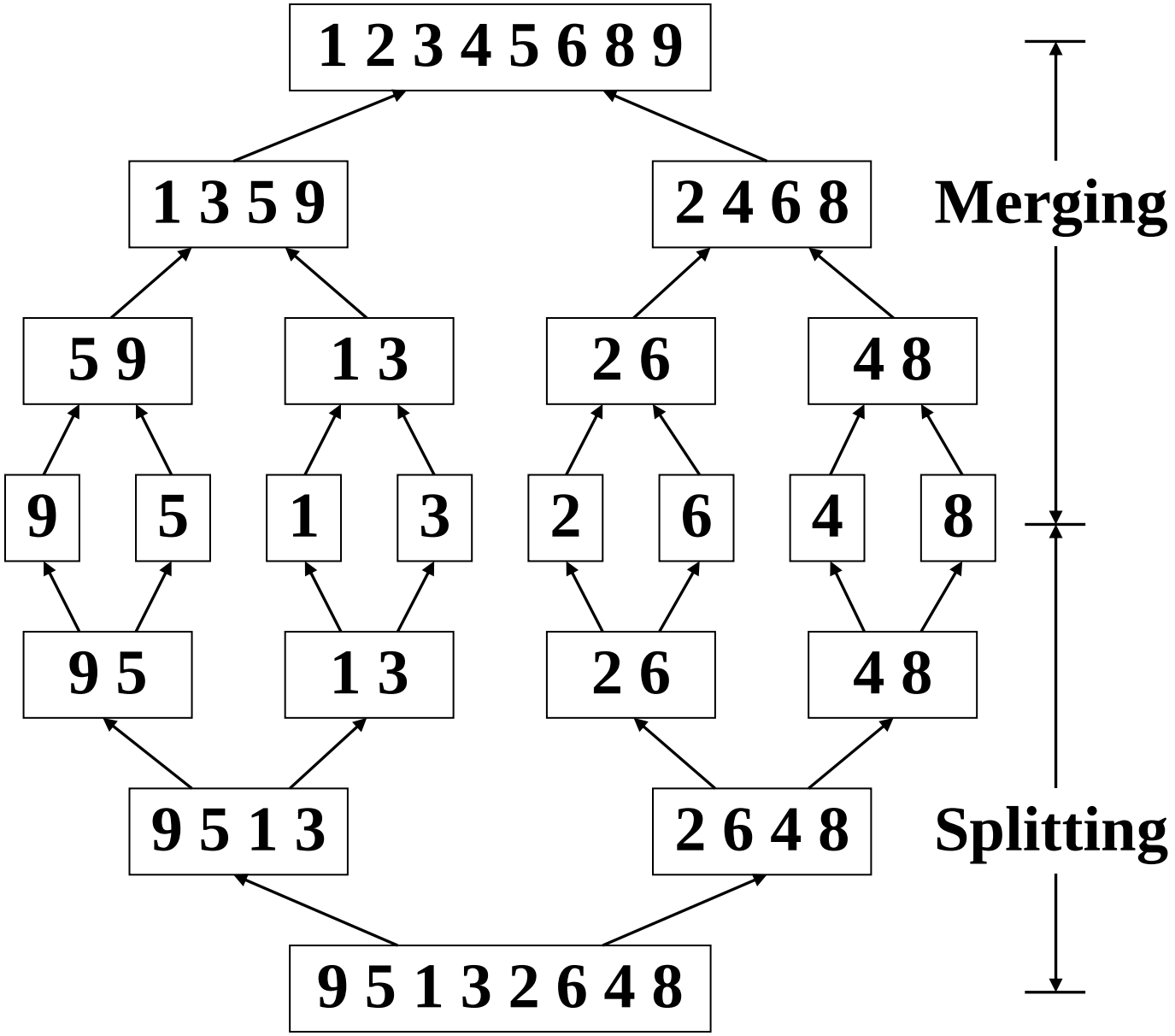
For merge sort the best, average and worst case execution time is $\Theta(n \lg n)$.

Recurrence tree



The height of this tree is $\log n$.
The execution time at each tree level is $\Theta(n)$.
So the total execution time is $\Theta(n \lg n)$

Call tree



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Analysis of divide and conquer algorithms

- The execution time can be described using a recurrence which describes the overall running time on a problem of size *n* in terms of running time on smaller inputs.
- Let *T*(*n*) be the execution time on a problem of size *n*. If *n* is sufficiently small, i.e. $n \leq c$ for some constant *c* the straightforward solution takes a constant time $\Theta(1)$.
- Let *D*(*n*) be the time to divide the problem into *a* sub-problems each of which is 1/*b* in size of the original.
- Let *C*(*n*) be the time to combine the solutions to the sub-problems into the solution to the original problem.

$$T(n) = \begin{cases} \Theta(1) & \text{if } n = c \\ aT(n/b) + D(n) + C(n) & \text{otherwise} \end{cases}$$

Growth of functions

- We can assign algorithms to complexity classes based on the asymptotic growth of their worst case execution time on the input size.

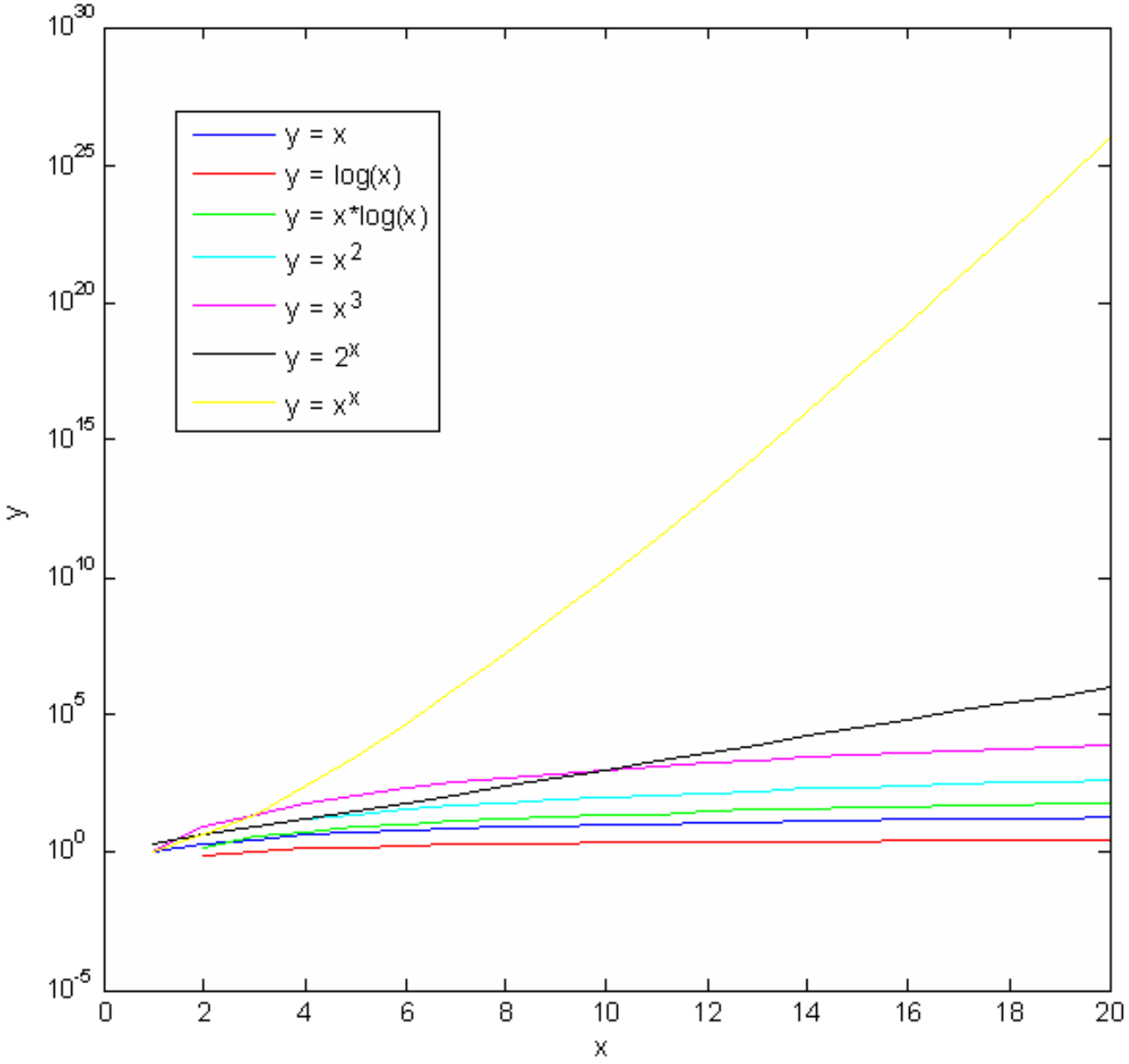
$$\Theta(g(n)) = \{f(n) | \exists c_1, c_2 \text{ and } n_0 > 0 \text{ such that } 0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \text{ for all } n \geq n_0\}$$

$$O(g(n)) = \{f(n) | \exists c \text{ and } n_0 > 0 \text{ such that } 0 \leq f(n) \leq c g(n) \text{ for all } n \geq n_0\}$$

$$\Omega(g(n)) = \{f(n) | \exists c \text{ and } n_0 > 0 \text{ such that } 0 \leq c g(n) \leq f(n) \text{ for all } n \geq n_0\}$$

$$f(n) \in \Theta(g(n)) \leftrightarrow (f(n) \in O(g(n))) \wedge (f(n) \in \Omega(g(n)))$$

Interpretation of growth of functions I



Value of *y* is shown on a logarithmic scale

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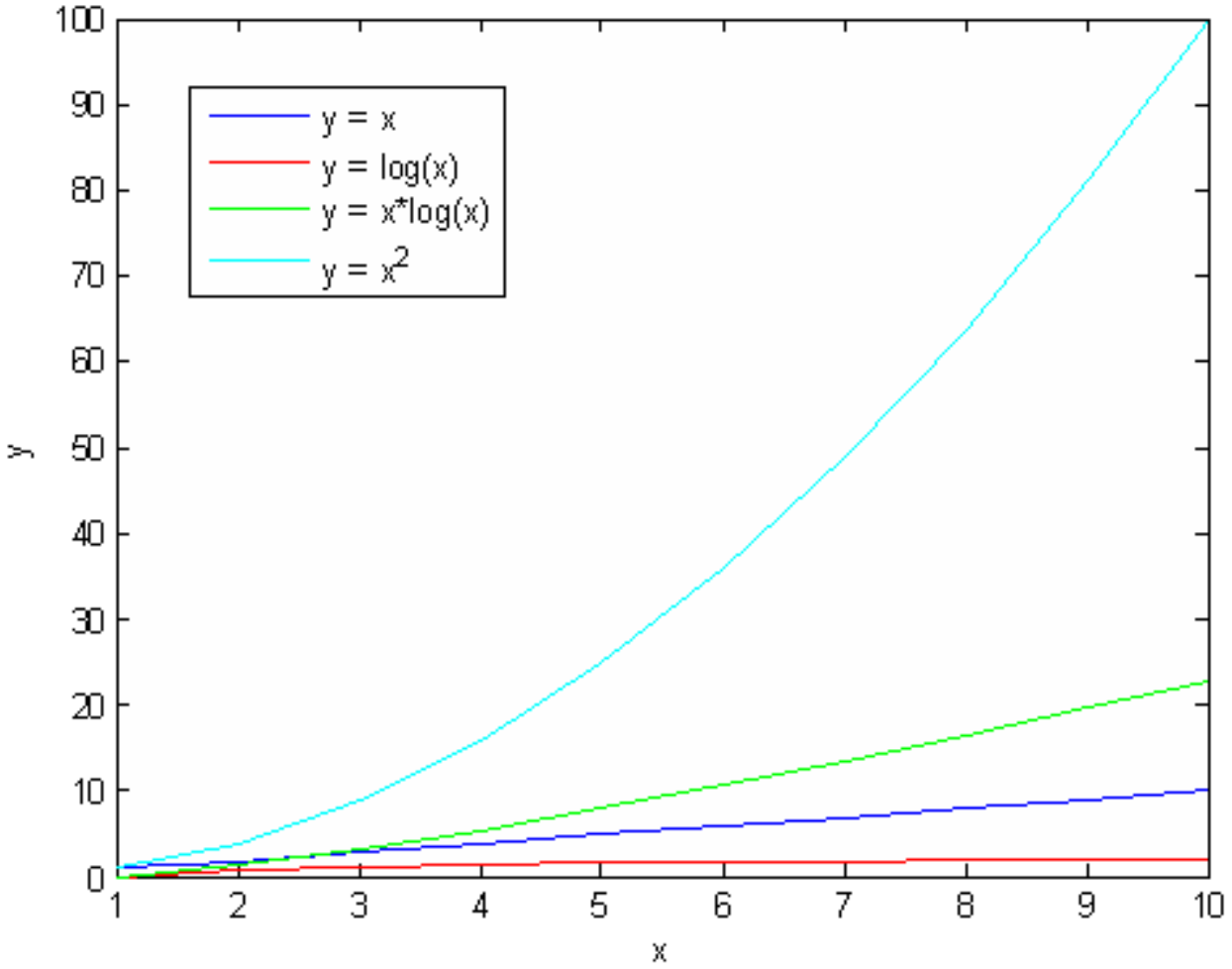
Interpretation of growth of functions II



Value of *y* is shown on a real scale

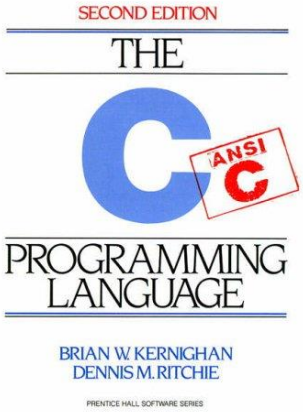
Other algorithms

- Selection sort: find the smallest element in $A[1...n]$, move it to the first position, find the second element in $A[2...n]$, move it to the second position, ... Evaluate the execution time as well.
- Binary search: search a value x in a sorted array $A[1...n]$ in time $\Theta(\lg n)$.
- Compute x^n with divide and conquer in time $\Theta(\lg n)$.
- Check if an array $A[1...n]$ contains at least two equal elements in time $\Theta(n \lg n)$.
- Consider an array $A[1...n]$ such that $A[i] \in \{1, \ldots, n\}$. Find an algorithm that checks non-destructively if A contains at least two equal elements and that takes execution time $\Theta(n)$ and $\Theta(1)$ memory.

World of Programming Languages

- Historical classification:
  - Machine code: 01111000 00000111
  - Assembler: MOV R0,#07
  - High-level languages: $a = b+1$
- Programming language paradigms:
  - Imperative (state-oriented): focused on “how?”
    - Procedural: C
    - Object-oriented: C++
  - Declarative (goal-oriented): focused on “what?”
    - Functional: Lisp, Haskell, ML, F# (a kind of ML), Erlang
    - Logic: Prolog
- Compiled vs Interpreted Languages: this distinction becomes more and more blurred today
  - Compiled: C, Assembler
  - Interpreted (scripting): Python, PHP, JavaScript
  - Partly compiled & partly interpreted: Java, C#

Brief Review of Standard C



- C is a *compiled language*.
- It is the basis for understanding more complex languages: C++, Java, C#.
- Our focus is *Standard C (ANSI C)* to assure program portability.
- A C program is a set of functions. A source file (module) may contain one or more functions. A C program is built from one or more C source files.
- There is a single function called *main*(). This defines the program's *entry point* – where the program's execution starts.

Functions and variables in C

- A function has:
  - a *signature* (also known as *prototype*) that states the types of arguments and the return type; it is given by a program statement called *declaration*.
  - a *definition* that defines the processing steps of the function (i.e. function body).
- The allowed prototypes of function *main*() are:

```
int main()
int main(int argc,char *argv[])
```

- Variables are either:
  - *local* (i.e. they are defined at the beginning of a block – a thing that starts with { and ends with }) or
  - *global* (i.e they are defined outside any block).

  Note that a function body is a block, so this applies to functions as well.

Control flow statements in C

- Sequencing:
  stmt `;` *stmt* `;` ...
- If-else:
  `if (`*expr*`)`
  stmt
  `else`
  stmt
- Switch:
  `switch (`*expr*`) {`
  `case` *const-expr* `:` *stmts*
  `case` *const-expr* `:` *stmts*
  ...
  `default:` *stmts*
  `}`
- While:
  `while (`*expr*`)`
  stmt
- Do-while:
  `do`
  stmt
  `while (`*expr*`);`
- For:
  `for (`$expr_1$ `;` $expr_2$ `;` $expr_3$`)`
  stmt
- Block:
  `{`
  stmt `;`
  stmt `;`
  ...
  stmt `;`
  `}`

Data types in C

- Scalar types:
  - arithmetic types
    - integral types: `char`, `short`, `int`, `long`
    - floating point types: `float`, `double`
  - pointer types: *T*`*`
- Aggregate types
  - `struct` type
  - `union` type
  - array type: *T*`[]`
  - function type: *T*`()`
- `void` type

- Note that integral types may be `signed` or `unsigned`.
- Note that arithmetic and `void` types are also known as *basic* or *primitive* types. They are the basic building blocks for the rest, *derived* types.

Brief overview of Python

- Python is an *interpreted language*.
- Python is characterized by:
  - Simple and readable syntax
  - Dynamic typing
  - High-level data types
- A Python program is a collection of functions and variables grouped into modules.
- A text file containing a sequence of Python statements is called a script and it has extension *.py*.
- A module is a *.py* file (or script) that usually contains a collection of functions, but also maybe other definitions.
- A Python program can be run:
  - Either by using the Python *interaction mode*
  - Or by running a script from the command line, under the Python control, i.e. in *script mode*.
- Python Software Foundation: https://www.python.org/

Variables and types in Python

- Python variables are *dynamically typed*.
- Each variable has a type based on its last assignment.
- Operators in expressions are *overloaded*.
- Types include:
  - Boolean (`bool`) with values `True` and `False`.
  - Numbers: integer (`int`) and real (floating-point) (`float`) numbers
  - Strings (`str`)
  - Lists: a sort of dynamically sized arrays (`list`)
  - Tuples: immutable arrays (`tuple`)
  - `NoneType`: a type with a single value `None`.
  - Dictionaries: a sort of maps or association lists.

Control flow statements in Python

- Sequencing:
  stmt
  ...
  stmt

- If-else:
  `if` *cond*:
      stmt
  `elif` *cond*:
      stmt
  ...
  `else`:
      stm

- While:
  `while` *cond*:
      stmt
      ...
      stmt

- For:
  `for` *var* **`in`** *range*:
      stmt
      ...
      Stmt

- Blocks use indentation with exactly 4 spaces.

Functions in Python

- They are defined using the `def` keyword.
- They can be placed in separate *.py* files and grouped in modules.
- Advanced *functional programming* techniques can be used in conjunction with functions.
- Functions can be defined as methods of classes using *object-oriented programming techniques*.

- Python can be considered as *multi-paradigm programming language*:
  - Imperative
  - Functional
  - Object-oriented